UNIVERSITY OF PENNSYLVANIA

CONVERGENCE OF SEQUENCES OF POSITIVE LINEAR FUNCTIONAL OPERATIONS

RICHARD PENNINGTON BAILEY

A DISSERTATION

IN MATHEMATICS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

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CONVERGENCE OF SEQUENCES OF POSITIVE LINEAR FUNCTIONAL OPERATIONS

BY R. P. BAILEY

Introduction. L. Fejér has recently called attention to the importance, in certain convergence problems of analysis, of a class of functionals to which he gives the name *positive operations.*[1] By his definition (loc. cit., p. 523), *a functional $U(x)$ defined over a set of functions $\{x(t)\}$, real-valued throughout a certain fundamental range $a \leqq t \leqq b$, is said to be positive, provided $U(x) \geqq 0$ whenever $x(t) \geqq 0$ in (a, b).* Sequences of operations of this kind often arise in the singular integral theory, in interpolation and in the theory of mechanical quadratures; in certain particular cases their convergence properties have been the object of much investigation. In his classical paper of 1909, Lebesgue gave the sequences of positive functionals which occur in the singular integral theory a special treatment, emphasizing again and again the simplicity of the reasoning involved, and their comparatively wide convergence properties.[2] At various times, many other writers have pointed out simplifications in a general theory which result from the hypothesis that certain sequences of functionals involved have the positive property.

It is our purpose, in the first part of this paper, to apply to the special case of the positive operations the very general theory which Hahn,[3] Banach[4] and others have developed for convergence problems involving sequences of linear functionals, with the object in view of establishing a set of theorems from which the particular theorems of Fejér, Lebesgue and others in the singular integral theory, mechanical quadratures and interpolation will be a matter of direct inference. The main discussion is divided into three parts. We take up first the question of the convergence of a sequence of positive linear functionals $\{U_n(x)\}$ $(n = 1, 2, \cdots)$ to the value of the function $x(t)(n \to \infty)$ at a certain fixed point $t = \tau$ of the fundamental interval. Sequences of this kind are familiar in interpolation and in the singular integral theory. In the second

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[1] L. Fejér, *On the infinite sequences arising in the theories of harmonic analysis, interpolation and mechanical quadratures,* Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 521–534.

[2] H. Lebesgue, *Sur les intégrales singulières,* Annales de Toulouse, (3), vol. 1(1909), pp. 25–117.

[3] H. Hahn, *Über Folgen linearer Operationen,* Monatshefte für Mathematik und Physik, vol. 32 (1922), pp. 3–88.

[4] S. Banach, *Théorie des Opérations Linéaires,* Monografje Matematyczne, Warsaw, 1932, pp. 122–130.

part we consider the convergence of the above sequence to a limit functional having the absolute continuity property of the Riemann integral and make applications to the problem of mechanical quadratures. In the third part we discuss an extension of the property of positiveness to more general operations, together with certain results on uniform convergence which can be obtained in this way.

The last section is devoted to a proof of the fact that one of the most important, although very special, types of positive operation, namely, the formulas of mechanical quadratures of Gauss' type, can have equal Côtes coefficients only in the well-known case of the trigonometric polynomials.

1. **Definition and properties of positive linear functional operations. Sequences of positive functionals.** Consider the function-space G whose elements $\{x(t)\}$ are the set of real-valued bounded functions of a real variable t defined over a closed finite interval[5] $a \leqq t \leqq b$. Two points of G are considered distinct if the corresponding functions differ at least at one point of (a, b). With the usual definitions of sum and scalar product, this set constitutes a linear or vectorial space. The zero element $x = \Theta$ is the point corresponding to the identically vanishing function $x(t) \equiv 0$. The point corresponding to the function $x(t) \equiv 1$ we denote by $x = I$. The distance (x, y) between two arbitrary points x and y of G we define to be $\underset{a \leqq t \leqq b}{\text{L.U.B.}} \mid x(t) - y(t) \mid$. With this distance-function, G becomes a metric space. It may be considered normal if we define the norm $\| x \|$ of a point x of G to be the non-negative number (x, Θ). If x and y belong to G and $x(t) \geqq y(t)$, we write $x \geqq y$.

A sequence of points $\{x_n\} (n = 1, 2, \cdots)$ of G is said to converge to the point x_0 of G, provided $\lim_{n \to \infty} \| x_n - x_0 \| = 0$. This clearly requires the uniform convergence of the sequence of functions $\{x_n(t)\}$ to $x_0(t)$.

A functional operation $U(x)$ defined over a vectorial subset E of G associates with each x of E a real number $U(x)$. This operation is said to be linear, if

$$\text{(i)}\quad U(x_1 + x_2) = U(x_1) + U(x_2) \qquad (x_1, x_2 \subset E),$$

$$\text{(ii)}\quad \lim_{n \to \infty} U(x_n) = U(x_0), \text{ whenever } \| x_n - x_0 \| \to 0 \qquad (\{x_n\}, x_0 \subset E);$$

that is, a linear operation is *additive* and *continuous*. Every linear operation is *homogeneous*, i.e., for an arbitrary real constant c and every x of E, $U(cx) = cU(x)$. Every linear operation is necessarily *bounded*, i.e., there exists a positive number $\mid U \mid_E = \underset{x \subset E, \| x \| \leqq 1}{\text{L.U.B.}} \mid U(x) \mid$, called the *norm* of the operation U over E, such that $\mid U(x) \mid \leqq \mid U \mid_E \| x \|$ for every x of E. As was stated above, U is said to be *positive* over E, if $U(x) \geqq 0$ whenever $x \geqq \Theta$. It follows at once, in virtue of (i), that here $x_1 \geqq x_2$ implies $U(x_1) \geqq U(x_2)$.

[5] This interval will be understood to be fixed throughout the paper unless the contrary is explicitly stated. In terminology, definitions and notations we follow S. Banach (loc. cit.).

The norm of a positive linear functional frequently has a very simple expression, as we show immediately.

LEMMA 1. *If the linear functional operation $U(x)$ is positive over a vectorial subset E of G which contains the point $x = I$, then $|U|_E = U(I)$.*

Evidently if $\|x\| \leqq 1$, $-I \leqq x \leqq I$, and therefore, since $U(x)$ is positive over E, $-U(I) \leqq U(x) \leqq U(I)$ and $|U(x)| \leqq U(I)$; whence

$$|U|_E = \underset{x \subset E, \|x\| \leqq 1}{\text{L.U.B.}} |U(x)| = U(I).$$

We shall need this result later on.

We now turn to the convergence of sequences of functionals. In general, for the convergence of a sequence $\{U_n(x)\}(n = 1, 2, \cdots)$ of linear functionals over a space E, it is sufficient to know (i) that it converges over some subspace H of E which is dense in E, and (ii) that the set of norms $\{|U_n|_E\}$ is bounded.[6] However, in the case of positive operations, these two hypotheses are sometimes redundant. Under certain conditions, (i) implies (ii), and hence (i) alone is sufficient. This point is brought out by

THEOREM 1. *Let E be a vectorial subset of G which contains an element of positive lower bound in (a, b). If a sequence $\{U_n(x)\}(n = 1, 2, \cdots)$ of positive linear functionals defined over E converges over a subset H dense in E to a linear*[7] *functional $U(x)$ defined over E, then $\lim_{n\to\infty} U_n(x) = U(x)$ for every x of E.*

It suffices to show that the set of norms $\{|U_n|_E\}(n = 1, 2, \cdots)$ is bounded. Since E contains an element of positive lower bound in (a, b), and H is dense in E, then H necessarily contains a positive element bounded away from zero, say $\xi(t) \geqq \rho > 0$. The function $\eta(t) = \rho^{-1}\xi(t)$ now has the property that $\eta(t) \geqq 1$, whence $-\eta \leqq x \leqq \eta$ for all x of E such that $\|x\| \leqq 1$. It follows, since the operations involved are positive, that $-U_n(\eta) \leqq U_n(x) \leqq U_n(\eta)$ and $|U_n(x)| \leqq U_n(\eta)$ ($\|x\| \leqq 1$; $n = 1, 2, \cdots$), whence $|U_n|_E \leqq U_n(\eta)$ ($n = 1, 2, \cdots$). This shows that the set of norms $\{|U_n|_E\}$ is bounded, since the sequence converges for $x = \xi$, and therefore for $x = \eta$.

That some restriction of the kind we have made on the subset E is necessary for the validity of the theorem can be seen from the following example. The functional

$$\sigma_n(x) = \frac{1}{2n\pi} \int_0^{2\pi} x(t) \left\{ \frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right\}^2 dt$$

represents the arithmetic mean of the first n partial sums, at the point $t = 0$, of the Fourier series associated with the integrable function $x(t)$, of period 2π. In particular,

[6] Banach, loc. cit., p. 123.

[7] Under very general conditions, the limit functional of a convergent sequence of linear functionals is necessarily linear itself. Cf. Banach, loc. cit., p. 122.

$$\sigma_n(I) = \frac{1}{2n\pi}\int_0^{2\pi}\left\{\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right\}^2 dt = 1 \qquad (n = 1, 2, \cdots). \tag{1}$$

Suppose now that our fundamental interval (a, b) is $(0, 2\pi)$. Let E be the set of continuous periodic functions $\{x(t)\}$ such that $x(0) = 0$, and let H denote that subset of E which consists of the trigonometric sums $\sum_{i=0}^{k} a_i \cos it + b_i \sin it$ $(k = 0, 1, 2, \cdots)$ which are zero at $t = 0$. H is dense in E by the Weierstrass theorem. Nevertheless, it can be shown that *the sequence of positive linear functionals*

$$U_n(x) \equiv n^{\frac{1}{2}}\sigma_n(x) \qquad (n = 1, 2, \cdots) \tag{2}$$

converges to the linear functional $U(x) = x(0) = 0$ *over* H *without converging at all points of* E. In fact, if x belongs to H, since the ordinary partial sums $S_n(x)$ of the first $2n + 1$ terms of the Fourier series associated with a trigonometric sum of order k are identical with the trigonometric sum itself, for $n \geqq k$, we readily conclude that in this case[8]

$$\lim_{n\to\infty} n^{1-\epsilon}\sigma_n(x) = 0$$

for any $\epsilon > 0$, and in particular for $\epsilon = \frac{1}{2}$. This shows that the sequence (2) converges over H. However, by (1),

$$|U_n|_E = \frac{n^{\frac{1}{2}}}{2\pi n}\int_0^{2\pi}\left\{\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right\}^2 dt = n^{\frac{1}{2}} \qquad (n = 1, 2, \cdots),$$

and therefore the sequence of norms $\{|U_n|_E\}$ is *unbounded.* Since E is complete in this case, we may conclude that there exists a point x of $E - H$ at which the sequence cannot converge to the value $x(0) = 0$, for it is well known that a sequence of linear functionals cannot converge over a complete vectorial space unless the norms are bounded in their set.[9] Theorem 1 does not apply here, since E does not contain an element bounded away from zero in $(0, 2\pi)$.

In what follows we shall find it convenient to refer to certain particular subsets of G as follows: P = the set of all polynomials of degrees $0, 1, 2, \cdots$, with real coefficients; T = the set of all trigonometric sums $\sum_{i=0}^{k} a_i \cos it + b_i \sin it$ $(k = 0, 1, 2, \cdots)$ with real coefficients; C = the set of all continuous functions; R = the set of all bounded R-integrable functions; L = the set of all bounded L-integrable functions; S = the set of step-functions with a finite

[8] $n^{1-\epsilon}\sigma_n = n^{-\epsilon}[S_0 + S_1 + \cdots + S_{n-1}] = n^{-\epsilon}[S_0 + S_1 + \cdots + S_{k-1}] = o(1)$, since $S_k = S_{k+1} = \cdots = S_{n-1} = 0$.

[9] Banach, loc. cit., p. 80, Th. 5.

number ($\geqq 0$) of steps; K = the set of all functions of bounded variation. It should be remarked that the same metric (that of G) is used throughout.

If E is an arbitrary subset of G, the symbols E_τ and E_p will be used to denote respectively the set of functions $\{x(t)\}$ of E which are continuous at a certain fixed point $t = \tau$ of (a, b), and those functions of E which are periodic, of period 2π. The symbol $E_{\tau p}$ will denote the set of functions of E having both properties.

With these notations, the subsets H and E of Theorem 1 may be taken to be P and C respectively, for C is evidently a vectorial subset of G containing positive elements bounded away from zero, in which the polynomials are dense by the Weierstrass theorem. Similarly, we might take $H = T$ and $E = C_p$, or $H = S$ and $E = K$.

2. **The limit-functional** $U(x) = x(\tau)$. When the convergence of a sequence of positive linear functionals $\{U_n(x)\}$ over a subset H dense in a vectorial subset E of G is known, Theorem 1 will, in general, enable us to draw conclusions about the convergence of the sequence at other points of E. However, in the particular case where the limit-functional is $U(x) = x(\tau)$, τ being some fixed point of (a, b), conclusions can be drawn about the convergence of the sequence over certain subsets of G from much weaker hypotheses. Theorem 2, below, illustrates this point. We first establish

LEMMA 2. *Let u, l denote, respectively, the upper and lower bounds of the bounded function $x(t)$ at the point $t = \tau$. Then there exist continuous functions $\xi_1(t)$ and $\xi_2(t)$ such that* (i) $\xi_1 \leqq x \leqq \xi_2$, (ii) $\xi_1(\tau) = l$, $\xi_2(\tau) = u$.

Let us construct, for example, the function ξ_2. We can suppose that $u < \| x \|$; if $u = \| x \|$, evidently $\xi_2(t) \equiv u$ is the function required. Let $\delta > 0$ be so chosen that $\underset{\tau-\delta \leqq t \leqq \tau+\delta}{\text{L.U.B.}} x(t) < \| x \|$, and denote by u_k $(k = 0, 1, 2, \cdots)$ the least upper bound of $x(t)$ in the interval $(\tau - \delta/2^k, \tau + \delta/2^k)$. Clearly $u_0 \geqq u_1 \geqq u_2 \geqq \cdots \geqq u_k \geqq \cdots$; $\lim_{k\to\infty} u_k = u$. Denote by P_k, Q_k respectively the points whose coördinates are $(\tau - \delta/2^{k+1}, u_k)$, $(\tau + \delta/2^{k+1}, u_k)$ and by P the point whose coördinates are (τ, u). Evidently $\lim_{k\to\infty} P_k = \lim_{k\to\infty} Q_k = P$; hence the polygonal line joining the points $(a, \| x \|)$, $(\tau - \delta, \| x \|)$, P_0, P_1, P_2, $\cdots$ in succession from the left, and the points $(b, \| x \|)$, $(\tau + \delta, \| x \|)$, Q_0, Q_1, Q_2, $\cdots$ in succession from the right, is continuous throughout (a, b). It defines a continuous function $\xi_2(t)$ which evidently meets the requirements of Lemma 2. The function $\xi_1(t)$ can be constructed in a similar manner. It should be noted that *if $x(t)$ is continuous at $t = \tau$, then $\xi_1(\tau) = \xi_2(\tau) = x(\tau)$.*

THEOREM 2. *Let E be a vectorial subset of G containing $C(C_p)$. If a sequence $\{U_n(x)\}(n = 1, 2, \cdots)$ of positive linear functionals defined over E converges over a subset H dense in $C(C_p)$ to the value $x(\tau)$ [τ a fixed point of (a, b)], then $\lim_{n\to\infty} U_n(x) = x(\tau)$ for every x of $E_\tau(E_{\tau p})$.*

Consider the first case; let x be any point of E. Since $x(t)$ is continuous at

$t = \tau$, we may conclude (by Lemma 2) that there exist functions $\xi_1(t)$, $\xi_2(t)$, belonging to C, such that

$$\xi_1 \leqq x \leqq \xi_2, \tag{3}$$

$$\xi_1(\tau) = x(\tau) = \xi_2(\tau). \tag{4}$$

Since the operations $\{U_n(x)\}$ are positive over E, (3) implies that

$$U_n(\xi_1) \leqq U_n(x) \leqq U_n(\xi_2) \qquad (n = 1, 2, \cdots). \tag{5}$$

By Theorem 1 both $\lim_{n\to\infty} U_n(\xi_1) = \xi_1(\tau)$ and $\lim_{n\to\infty} U_n(\xi_2) = \xi_2(\tau)$ exist, and hence by (5) $\xi_1(\tau) \leqq \overline{\lim_{n\to\infty}} U_n(x) \leqq \xi_2(\tau)$. This proves our statement, in virtue of (4).

When C and E_τ are replaced by C_p and $E_{\tau p}$, respectively, ξ_1 and ξ_2 must be so chosen as to be periodic, of period 2π, as well as continuous. Though Lemma 2 does not affirm the existence of such functions, it is clear that methods analogous to those used in its proof will furnish the required construction, if we use the fact that $x(t)$ (belonging to $E_{\tau p}$) is now periodic, of period 2π, as well as continuous at $t = \tau$.

The question naturally arises whether a sequence of positive linear functionals $\{U_n(x)\}$ satisfying the hypotheses of Theorem 2 will not further converge to the value $\frac{1}{2}[x(\tau + 0) + x(\tau - 0)]$ for those functions $\{x(t)\}$ of E for which this functional is defined. This cannot be the case, in general, as one sees easily by considering the special sequence

$$U_n(x) \equiv x(\tau + 0) \qquad (n = 1, 2, \cdots). \tag{6}$$

The sequence (6) is defined, for instance, over K, and converges to the value $x(\tau)$ over K_τ by identity, but cannot converge to the value $\frac{1}{2}[x(\tau + 0) + x(\tau - 0)]$ at any point of K where $x(\tau + 0) \neq x(\tau - 0)$.

The following theorem gives a sufficient condition for convergence of the type described above.

THEOREM 3. *Let E be a vectorial subset of G containing K. If a sequence $\{U_n(x)\}(n = 1, 2, \cdots)$ of positive linear functionals defined over E converges over S to the value $\frac{1}{2}[x(\tau + 0) + x(\tau - 0)]$, then $\lim_{n\to\infty} U_n(x) = \frac{1}{2}[x(\tau + 0) + x(\tau - 0)]$ for every x of E which has a discontinuity of the first kind at $t = \tau$.*

The proof follows the same lines as that of Theorem 2. Since S is dense in K, Theorem 1 assures the convergence of the sequence to the required value over K. We now have only to construct, by the methods of Lemma 2, two functions ξ_1, ξ_2 of K such that $\xi_1 \leqq x \leqq \xi_2$ and $\xi_1(\tau + 0) = \xi_2(\tau + 0) = x(\tau + 0)$, $\xi_1(\tau - 0) = \xi_2(\tau - 0) = x(\tau - 0)$. The convergence of the sequence for the element x (of which we assume only that it has at most a discontinuity of the first kind at $t = \tau$) follows as in Theorem 2.

We may mention in passing an obvious extension of the theorems of this section which can be proved by means of Lemma 2.

THEOREM 4. *Let u_x, l_x denote respectively the upper and lower bounds of $x(t)$*

at the point $t = \tau$. If a sequence $\{U_n(x)\}(n = 1, 2, \cdots)$ of positive linear functionals defined over a vectorial subset E of G which contains C converges over a subset H dense in C to the value $x(\tau)$, then $l_x \leqq \underline{\lim}_{n\to\infty} U_n(x) \leqq \overline{\lim}_{n\to\infty} U_n(x) \leqq u_x$ at every point of E.

In Theorem 2 we may set $E = L$, $H = S_\tau$ (the set of step-functions with a finite number of steps which are continuous at $t = \tau$), and

$$U_n(x) \equiv \int_a^b x(t)\varphi(t - \tau, n)\, dt,$$

where, for all $n = 1, 2, \cdots, \varphi(\alpha, n)$ is a bounded, non-negative, L-integrable function of α. These substitutions are admissible, and our conclusion in this special case is that the convergence of the given sequence (a so-called "singular" integral) to the value $x(\tau)$ over S_τ implies convergence to the value $x(\tau)$ for every bounded L-integrable function $x(t)$ continuous at $t = \tau$. This result was obtained for the first time by Lebesgue.[10]

The choice of the set H of Theorem 2 depends, in general, upon the nature of the functionals $\{U_n(x)\}$. For instance, in discussing the convergence properties of Fejér's integral, we might take $H = T$ rather than $H = S_\tau$ (as in the theorem of Lebesgue), for the convergence of the Fejér sequence over T follows directly from its definition, while the question of its convergence over S_τ requires further investigation.

In a similar manner, Theorem 3 and Theorem 4 can be interpreted so as to give certain results of Lebesgue and others on the behavior of singular integrals for functions discontinuous[11] at $t = \tau$.

The following is an application of Theorem 2 in a situation where the singular integral theory cannot be used directly, i.e., interpolation.[12] Let

$$\begin{array}{ccccccc}
t_{11} & & & & & & \\
t_{12} & t_{22} & & & & & \\
t_{13} & t_{23} & t_{33} & & & & \\
\cdots & \cdots & \cdots & & & & \\
t_{1n} & t_{2n} & t_{3n} & \cdots & t_{nn} & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & &
\end{array}$$

[10] Lebesgue, loc. cit., p. 75. The hypothesis of Lebesgue requires the convergence of the given sequence of integrals over what is known as the base of Hamel of the set S_τ (i.e., a linearly independent subset of S_τ upon which every other element of S_τ is dependent) and therefore is equivalent to an assumption of convergence over S_τ itself. In the same way, instead of assuming that a certain sequence of linear functionals converges for all polynomials, we may assume convergence only over the set $1, t, t^2, \cdots$; the two assumptions are equivalent, since each is an immediate inference of the other.

[11] Lebesgue, loc. cit., p. 78. See also Hobson, *Theory of the Functions of a Real Variable*, vol. 2(1926), p. 456, where a special case of Theorem 4 is discussed.

[12] H. Hahn, *Über Interpolation*, Mathematische Zeitschrift, vol. 1 (1918), pp. 115–43, first suggested an attack on the general problem of interpolation from the viewpoint of the theory of linear operations; certain of the results of his paper *Über Folgen linearer Operationen* (loc. cit.) can be applied to the problem of interpolation to continuous functions.

be a triangular sequence of real numbers such that $a \leqq t_{1n} < t_{2n} < \cdots < t_{nn} \leqq b$ $(n = 1, 2, \cdots)$, and let $\{h_{in}(t)\}$ $(i = 1, 2, \cdots, n; n = 1, 2, \cdots)$ be a set of continuous functions defined on (a, b), having the property $h_{in}(t_{kn}) = \delta_{ik}$, where δ is the Kronecker delta. The expression

$$H_n(t) = \sum_{i=1}^{n} x(t_{in})\, h_{in}(t) \qquad (n = 1, 2, \cdots) \tag{7}$$

can then be considered a formula of interpolation to the function $x(t)$ in the interval (a, b). Let us suppose further that the fundamental functions $\{h_{in}(t)\}$ are positive[13] throughout (a, b). Theorem 2 enables us to conclude at once that the convergence of the sequence of positive linear functionals $U_n(x) \equiv H_n(\tau)$ $(n = 1, 2, \cdots)$ to the value $x(\tau)$ over P or T will imply the convergence of the given sequence over G_τ or $G_{\tau p}$. Hence the formula (7) converges, in this case, to the value of the function $x(t)$ at every point of continuity. Once more we have an illustration of the importance of the very general theorems of this section: the convergence properties of the formula (7) follow directly from Theorem 2 without any further considerations.

3. **Absolutely continuous limit-functionals.** We now pass to considerations which have important applications in the theory of mechanical quadratures. In this section attention will be confined to the set R of bounded functions integrable in the sense of Riemann. We introduce first some notations which materially simplify the discussion. Let J denote a finite set of disjoint intervals contained in (a, b), and J' its complement. With each x of R associate an element x_J of R as follows: $x_J(t) = x(t)$ or 0 according as t belongs to J or J'. With each operation $U(x)$ defined over R associate an operation $U^J(x)$ as follows: $U^J(x) = U(x_J)$. Clearly, if $U(x)$ is a linear functional over R, $U^J(x)$ is also; if $U(x)$ is positive over R, $U^J(x)$ has the same property. Evidently $U(x) = U^J(x) + U^{J'}(x)$ for every x of R.

The following is a generalization of theorems due to L. Fejér[14] and G. Pólya.[15]

THEOREM 5. *Let $\alpha(t)$ be absolutely continuous on (a, b). If a sequence $\{U_n(x)\}$ $(n = 1, 2, \cdots)$ of positive linear functionals defined over R converges over P to the functional $U(x) = \int_a^b x(t)\, d\alpha(t)$, then*[16] $\lim_{n\to\infty} U_n(x) = U(x)$ *for every x of R.*

[13] Interpolation formulas of this type have been studied by L. Fejér, *Über Interpolation*, Göttinger Nachrichten, vol. (1916), pp. 66–91, and D. Jackson, *The Theory of Approximation*, Colloquium Publications of American Mathematical Society, vol. 11 (1930), pp. 142–148. See also in this connection J. Shohat, *On interpolation*, Annals of Mathematics, vol. 34 (1933), pp. 130–146, and G. Szegö, *Interpolationspolynome*, Mathematische Zeitschrift, vol. 35 (1932), pp. 579–602.

[14] L. Fejér, *Mechanische Quadraturen mit positiven Côtesschen Zahlen*, Mathematische Zeitschrift, vol. 37 (1933), pp. 287–309.

[15] G. Pólya, *Über die Konvergenz von Quadraturverfahren*, Mathematische Zeitschrift, vol. 37 (1933), pp. 264–286.

[16] The integral is to be taken in the Riemann-Stieltjes sense. It is defined over R, since $\alpha(t)$ is absolutely continuous.

Under our hypothesis, and by Theorem 1, $\lim_{n\to\infty} U_n(x) = U(x)$ for every continuous function $x(t)$. Hence $U(x)$ is necessarily positive over C, being the limit, when $x \geqq \Theta$, of a sequence of non-negative numbers. We readily conclude that $\alpha(t)$ is monotone non-decreasing,[17] and from this it follows that $U(x)$ is positive over R.

Suppose now J is any finite set of disjoint closed intervals. We will show that $\lim_{n\to\infty} U_n^J(I) = U^J(I)$. Evidently, it will be sufficient to consider the case where J consists of a single closed interval (λ, μ), where $a < \lambda < \mu < b$. The proof can then be extended to the most general case by the use of the homogeneous-additive property of the linear functional. Let $x(t) = 1$ or 0 according as t belongs to J or J'. Then $U_n(x) = U_n^J(I)$ $(n = 1, 2, \cdots)$. We will show that $\lim_{n\to\infty} U_n(x)$ exists and has the value $U(x) = U^J(I)$. Let $\delta > 0$ be taken so small that $a \leqq \lambda - \delta$, $\lambda + \delta < \mu - \delta$ and $\mu + \delta \leqq b$. Define auxiliary functions $y(t)$ and $z(t)$, belonging to C, as follows:

$$y(t) = \begin{cases} 0 & a \leqq t \leqq \lambda - \delta \\ 1 & \lambda \leqq t \leqq \mu \\ 0 & \mu + \delta \leqq t \leqq b \\ \text{linear elsewhere,} & \end{cases} \qquad z(t) = \begin{cases} 0 & a \leqq t \leqq \lambda \\ 1 & \lambda + \delta \leqq t \leqq \mu - \delta \\ 0 & \mu \leqq t \leqq b \\ \text{linear elsewhere.} & \end{cases}$$

Now, $y \geqq x \geqq z$. Therefore, since the operations $\{U_n(x)\}$ and the operation $U(x)$ are positive over R,

$$U_n(y) \geqq U_n(x) \geqq U_n(z) \qquad (n = 1, 2, \cdots), \tag{8}$$

$$U(y) \geqq U(x) \geqq U(z). \tag{9}$$

From (8), $U(y) \geqq \overline{\lim_{n\to\infty}} U_n(x) \geqq U(z)$, for y and z are elements of C, whence $\lim_{n\to\infty} U_n(y) = U(y)$ and $\lim_{n\to\infty} U_n(z) = U(z)$. Finally, since

$$| U(y) - U(z) | = \left| \int_a^b [y(t) - z(t)]\, d\alpha(t) \right| \leqq 2V_\delta,$$

where V_δ represents the total variation of $\alpha(t)$ over the intervals $(\lambda - \delta, \lambda + \delta)$ and $(\mu - \delta, \mu + \delta)$, and V_δ is known to approach zero with δ, we conclude that $\lim_{n\to\infty} U_n(x)$ exists, and by (9) $\lim_{n\to\infty} U_n(x) = U(x)$. This proves our statement.

With this, we are in a position to establish the main theorem. Let x now be any element of R. Being given $\epsilon > 0$, by the criterion for Riemann integrability, it is possible to fit a net D on the interval (a, b) such that the sum of the lengths of those intervals of D is $< \epsilon$, in which the fluctuation of $x(t)$ is $\geqq \epsilon$. Let $\xi(t)$ be that continuous function which coincides with $x(t)$ at the end points of the intervals of D and is linear in the interior of each of these intervals. If we

[17] If we assume the contrary, there is no difficulty in constructing a non-negative continuous function $x(t)$ such that $\int_a^b x(t)\, d\alpha(t) < 0$.

denote by J the interval-set, throughout which the fluctuation of $x(t)$ is $\geqq \epsilon$, evidently $| x(t) - \xi(t) | < \epsilon$ for all values of t in the intervals J' complementary to J. We may take J to be closed. By our construction $m(J) < \epsilon$. Now

$$| U(x) - U_n(x) | \equiv | U_n(\xi - x) + U(x - \xi) + U(\xi) - U_n(\xi) |$$
$$\leqq | U_n(\xi - x) | + | U(x - \xi) | + | U(\xi) - U_n(\xi) |.$$

Using the fact that $| U_n^J |_R = U_n^J(I)$ (by Lemma 1), and since $| x - \xi | \leqq 2 | x |$, we conclude in succession,

$$| U_n(\xi - x) | \leqq | U_n^J(\xi - x) | + | U_n[(\xi - x)_{J'}] |$$
$$\leqq 2 \| x \| \, | U_n^J |_R + \epsilon \, | U_n |_R = 2 \| x \| \, U_n^J(I) + \epsilon \, U_n(I),$$
$$| U(x - \xi) | \leqq | U^J(x - \xi) | + | U[(x - \xi)_{J'}] |$$
$$\leqq 2 \| x \| \, | U^J |_R + \epsilon \, | U |_R = 2 \| x \| \, U^J(I) + \epsilon U(I);$$

and since ξ belonging to C implies $\lim_{n \to \infty} | U(\xi) - U_n(\xi) | = 0$, evidently

$$\overline{\lim_{n \to \infty}} | U(x) - U_n(x) | \leqq 4 \| x \| \, U^J(I) + 2\epsilon U(I), \tag{10}$$

for we have shown that $\lim_{n \to \infty} U_n^J(I) = U^J(I)$. Since $U^J(I) = \int_J d\alpha(t)$ represents simply the variation of the absolutely continuous function $\alpha(t)$ over the set J, and $m(J) < \epsilon$, it is evident that the quantity on the right hand side of (10) can be made arbitrarily small by choosing ϵ sufficiently small, and Theorem 5 is established.

Examination of the proof of Theorem 5 will show that the only property of the limit-functional $U(x)$ really needed in the demonstration (in addition to linearity) is the absolute continuity property which it has in common with the ordinary Riemann integral. This property may be characterized as follows: a linear functional $U(x)$, defined over R, will be said to be *absolutely continuous*, provided that for every x of R, $\epsilon > 0$ arbitrarily given implies the existence of a positive number $\delta = \delta(\epsilon, x)$ such that if J is any finite set of disjoint intervals contained in (a, b), and $m(J) < \delta$, then[18] $| U^J(x) | < \epsilon$. The question arises whether we cannot obtain greater generality by replacing $\int_a^b x(t) \, d\alpha(t)$ with a more general absolutely continuous limit-functional. The fact is, *every linear functional $U(x)$ defined over R which possesses the absolute continuity property above described is necessarily of the form* $\int_a^b x(t) \, d\alpha(t)$, *where $\alpha(t)$ is an absolutely continuous function.* This statement may be proved as follows. With the methods which Banach (loc. cit., p. 59) has used to establish the well-known theorem of

[18] In effect, we have called $U(x)$ absolutely continuous if, for fixed x, $U^J(x)$ is an absolutely continuous interval-set function $\Phi_x(J)$ in the sense of de La Vallée-Poussin, *Intégrales de Lebesgue*, Borel Collection, 1916, p. 57.

F. Riesz[19] on the general form of linear functionals defined over C-space, there is no difficulty in showing that every absolutely continuous linear functional $U(x)$ defined over R is of the form $\int_a^b x(t)d\alpha(t)$ ($\alpha(t)$ absolutely continuous) over C-space. We have to show that the given expression is the only extension of this functional to R-space which has the required property. For this we need

LEMMA 3. *If a linear functional $U(x)$ has the absolute continuity property over R, $| U^J |_R \to 0$ uniformly with $m(J)$ (i.e., $U(x)$ has this property uniformly over every bounded subset of R).*

We wish to show that $\epsilon > 0$ being given arbitrarily there exists a $\delta = \delta(\epsilon)$, such that if J is any finite set of disjoint intervals, and $m(J) < \delta$, then $| U^J(x) | < \epsilon$ for every x of R with $\| x \| \leqq 1$. Assume the contrary. Then there exists a $\rho > 0$, such that no matter how small $\delta > 0$ be taken, there is an $x = x_\delta$ (of norm $\leqq 1$) and a set $J = J_\delta$, with $m(J) < \delta$, such that $U^J(x) \geqq \rho$. That is, there exists a sequence of elements $\{x_n\}$ ($n = 1, 2, \cdots$) of R of norm $\leqq 1$, and a sequence of sets J_n ($n = 1, 2, \cdots$) with the property $\lim_{n\to\infty} m(J_n) = 0$, such that

$$U^{J_n}(x_n) \geqq \rho \qquad (n = 1, 2, \cdots). \tag{11}$$

Let N be a given positive integer, and let $\delta_k > 0$ ($k = 1, 2, \cdots$) be so chosen that $| U^J(x_k) | < 1/2^k$ for every J such that $m(J) < \delta_k$. This is possible on account of the absolute continuity property of $U(x)$. Since $m(J_n)$ approaches zero with $1/n$, it is possible to select a set of indices $n = n_1, n_2, \cdots, n_N$ such that the part which J_{n_k} ($k = 1, 2, \cdots, N$) has in common with the sets which follow (namely, $J_{n_{k+1}}, J_{n_{k+2}}, \cdots, J_{n_N}$) is of measure $< \delta_{n_k}$; let us call the sequence of sets obtained by removing this part $L_1, L_2, \cdots, L_N$. It follows that no two of the sets $\{L_k\}$ can have a point in common; each L_k is a subset of J_{n_k} and differs in measure from that of J_{n_k} by less than δ_{n_k}. Suppose now η is that element of R which is zero throughout the complement of $(L_1 + L_2 + \cdots + L_N)$, and in the set L_k ($k = 1, 2, \cdots, N$) coincides with x_{n_k}. We have:

$$U(\eta) = \sum_{k=1}^{N} U^{L_k}(x_{n_k}) = \sum_{k=1}^{N} U^{J_{n_k}}(x_{n_k}) - \sum_{k=1}^{N} U^{J_{n_k}-L_k}(x_{n_k}),$$

$$\geqq N\rho - \sum_{k=1}^{N} \frac{1}{2^{n_k}} \geqq N\rho - 1,$$

in virtue of (11) and the fact that $| U^J(x_{n_k}) | < 1/2^{n_k}$ for every J such that $m(J) < \delta_{n_k}$(in particular, $J = J_{n_k} - L_k$); since $\| \eta \| \leqq 1$, and N may be chosen arbitrarily large, this contradicts the boundedness of the linear functional $U(x)$, and the lemma is proved.

[19] F. Riesz, Annales de l'École Normale Supérieure, (3), vol. 31 (1914), pp. 9–14. This theorem has been extended to the class of bounded functions with at most discontinuities of the first kind by H. S. Kaltenborn, Bulletin of the American Mathematical Society, vol. 40(1934), pp. 702–708, and by T. H. Hildebrandt, Transactions of the American Mathematical Society, vol. 36(1934), pp. 868–875, to the class of all bounded measurable functions.

With this lemma, our original statement can easily be proved. Let x be any element of R. As we have seen in the proof of Theorem 5, there exists an element ξ of C, and a finite set J of disjoint intervals with $m(J) < \epsilon$, such that $|x(t) - \xi(t)| < \epsilon$ (t in $c(J) = J'$), $\|x - \xi\| \leqq 2\|x\|$. Then, since $U(\xi) = \int_a^b \xi(t)d\alpha(t)$,

$$\left|U(x) - \int_a^b x(t)\,d\alpha(t)\right| = \left|U(x-\xi) - \int_a^b [x(t) - \xi(t)]\,d\alpha(t)\right|$$

$$\leqq |U(x-\xi)| + \left|\int_a^b [x(t) - \xi(t)]\,d\alpha(t)\right|$$

$$\leqq 2\|x\|\,|U^J|_R + \epsilon\,|U|_R + 2\|x\| \int_J d\alpha(t) + \epsilon V_a,$$

where $U(x - \xi) = U^J(x - \xi) + U^{J'}(x - \xi)$, etc., and where V_a denotes the total variation of $\alpha(t)$ over (a, b). Since, by Lemma 3, $|U^J|_R \to 0$ uniformly with $m(J)$, as $\epsilon \to 0$, and since $\alpha(t)$ is absolutely continuous, the right member of this inequality can be made arbitrarily small by choosing ϵ sufficiently small. Hence the left member must be zero, and our statement is proved.

As an application of Theorem 5, consider the formulas of mechanical quadratures of Gauss' type

$$\int_a^b x(t)p(t)\,dt = \sum_{i=1}^{n} H_{in}x(t_{in}) + R_n(x) \qquad (n = 1, 2, \cdots),$$

where $p(t)$ is non-negative and summable in (a, b), determined by the conditions

$$R_n(t^k) \equiv 0 \qquad (k = 0, 1, 2, \cdots, 2n - 1;\ n = 1, 2, \cdots). \tag{12}$$

It can be shown that

$$H_{in} = \int_a^b \left[\frac{\Phi_n(t)}{(t - t_{in})\Phi_n'(t_{in})}\right]^2 p(t)\,dt \qquad (i = 1, 2, \cdots, n;\ n = 1, 2, \cdots)$$

$$\left[\Phi_n(t) \equiv (t - t_{1n})(t - t_{2n}) \cdots (t - t_{nn}),\ \int_a^b \Phi_n\Phi_m p(t)\,dt = 0\ (m \neq n)\right],$$

whence[20] $H_{in} > 0$ for all i and n. Putting $U_n(x) \equiv \sum_{i=1}^{n} H_{in}x(t_{in})$, $U(x) \equiv \int_a^b x(t)\,d\alpha(t)$, where $\alpha(t) \equiv \int_a^t p(t)\,dt$, and applying Theorem 5, we conclude that the formulas of Gauss' type converge for all bounded R-integrable functions $\{x(t)\}$, since convergence over P is assured by (12). The operations $\{U_n(x)\}$ are positive, since $H_{in} > 0$ ($i = 1, 2, \cdots, n;\ n = 1, 2, \cdots$).

[20] J. Shohat, *Théorie générale des polynomes orthogonaux de Tchebichef*, Mémorial des Sciences Mathématiques, LXVI (1934), p. 15.

In the paper referred to at the beginning of this section,[14] Fejér showed that *any* mechanical quadratures formulas of the type

$$\int_{-1}^{+1} x(t)\,dt = \sum_{i=1}^{n} H_{in}x(t_{in}) + R_n(x) \qquad (n = 1, 2, \cdots) \tag{13}$$

based on the Lagrange interpolation formula (i.e., $R_n(t^k) = 0$, $k = 0, 1, 2, \cdots, n-1$) must converge for all R-integrable functions $x(t)$ if $H_{in} \geqq 0$. Pólya[15] has also given theorems from which the same conclusion can be derived. This result follows at once from Theorem 5.

4. **Non-functional positive operations. Uniform convergence.** We have shown, in the preceding pages, how the characterization of a certain type of linear functional as positive proves useful in discussing the convergence properties of sequences of such functionals; we shall now try to indicate how this characterization can be extended to include certain types of linear operations which are not necessarily functional. Consider, for example, an operation $U(x)$ defined over a vectorial subset E of G which associates with every x of E an element $y \equiv U(x)$ of some subset E_1 of G. E_1 is called the *range* or *contra-domain* of the operation $U(x)$. We make the following definition: $U(x)$ *is positive over* E, *provided* $x \geqq \Theta$ *implies* $y \geqq \Theta$. It is clear that we have here a direct extension of the definition of positiveness given previously for functionals. We have simply replaced the contra-domain r of real numbers by another normal vectorial space in which the symbol $\geqq$ is defined.

The statements made in §1 concerning the definition and properties of linear functionals defined over G apply without exception to linear operations with domain E and contra-domain E_1. In general, we use the same notations as in the previous case, though it should be noted that here the norm of $U(x)$ (as an element of G) must be written $\| U(x) \|$. An analogue to Lemma 1 follows at once.

LEMMA 4. *If the linear operation* $U(x)$, *defined over a vectorial subset* E *of* G *which contains* $x = I$, *and with contra-domain lying in* G, *is positive over* E, *then* $| U |_E = \| U(I) \|$.

Exactly the same argument as that given in the proof of Lemma 1 will suffice to establish Lemma 4, if we note that in G, as well as with real numbers, $-x \leqq y \leqq x$ implies $\| y \| \leqq \| x \|$.

Similarly, Theorem 1 has its analogue, which we can state as follows:

THEOREM 6. *Let* E *be a vectorial subset of* G *containing an element of positive lower bound in* (a, b). *If a sequence* $\{U_n(x)\}$ $(n = 1, 2, \cdots)$ *of positive linear operations defined over* E, *and with contra-domain lying in* G, *converges over a subset* H *dense in* E *to a linear operation* $U(x)$ *defined over* E, *with contra-domain in* G, *then*[21] $\lim_{n\to\infty} U_n(x) = U(x)$ *for every* x *of* E.

[21] Where previously we dealt with sequences of numbers, here it is a question of the convergence of a sequence of *functions* $\{U_n(x)\}$. This convergence is uniform, in view of the metric of G.

Again, the proof follows exactly the same lines as that of Theorem 1. We see without difficulty that the sequence converges for an element $\eta(t)$ of E such that $-\eta \leqq x \leqq \eta$ for all x of E with $\| x \| \leqq 1$. It follows that $| U_n |_E \leqq \| U_n(\eta) \|$ $(n = 1, 2, \cdots)$. This proves the theorem, for it shows that the set of norms $\{| U_n |_E\}$ is bounded. From this follows the convergence of the sequence to $U(x)$ at all points[22] of E.

As an application of Theorem 6, consider the integral of Fejér. Let

$$U_n(x) \equiv y_n(t) \equiv \frac{1}{2n\pi} \int_0^{2\pi} x(s) \left\{ \frac{\sin \frac{1}{2} n(s-t)}{\sin \frac{1}{2}(s-t)} \right\}^2 ds \qquad (n = 1, 2, \cdots). \tag{14}$$

The sequence (14) is a sequence of positive operations defined over C_p, $[(a, b) \equiv (0, 2\pi)]$, with contra-domain lying in C_p, which converges to the operation $U(x) = x(t)$ over T; by Theorem 6 the sequence of functions $\{y_n(t)\}$ converges uniformly over $(0, 2\pi)$ to the value $x(t)$ for every function $x(t)$ continuous in $(0, 2\pi)$ and of period 2π.

We can prove further that the convergence of the Fejér integral to the value $x(t)$ is uniform over any subinterval of $(0, 2\pi)$ in which $x(t)$ is continuous. Results of this kind can evidently be obtained for any singular integral or interpolation formula of positive character.

It is clear that the idea of positiveness in connection with linear operations can be extended to include operations defined over the most general abstract space of the normal vectorial type, with elements of any nature whatsoever, provided only some convenient meaning be assigned to the symbol $\geqq$ in domain and contra-domain. Whether such a classification of linear operations will be useful will depend to a great extent upon the type of space. The author hopes, at some future date, to extend the idea to the Hilbert spaces, with applications to the theory of integral equations in view.

5. **Coincidence of the formulas of mechanical quadratures of Gauss' type and of Tchebichef's type.** In §3 we called attention to the convergence properties of certain mechanical quadratures formulas of Gauss' type. In this section we consider a special problem connected with such formulas. Let $\psi(t)$ be a function bounded and non-decreasing, with infinitely many points of increase, over an interval (a, b) finite or infinite, and such that all moments $\alpha_n = \int_a^b t^n d\psi(t)$ $(n = 0, 1, 2, \cdots)$ exist, with $\alpha_0 > 0$. It is known that there exists an infinite sequence of orthogonal Tchebichef polynomials

$$\Phi_n(t) = \prod_{i=1}^{n} (t - t_{in}) \equiv t^n + p_{1n} t^{n-1} + \cdots + p_{nn},$$

[22] Banach, loc. cit., p. 79.

with all roots real, distinct, and lying in (a, b), determined by the conditions[23] $\int_a^b t^k \Phi_n(t) d\psi(t) = 0$ $(k = 0, 1, 2, \cdots, n-1; n = 1, 2, \cdots)$. In order to avoid trivial equivalences, we assume $\psi(a) = 0$ and $\int_a^{a+h} d\psi(t) \int_{b-h}^b d\psi(t) > 0$ for $h > 0$. Under these conditions $\psi(t)$ will be called a characteristic function in (a, b). Two characteristic functions will be considered distinct only if they differ at a point of continuity.

Let $x(t)$ be such that $\int_a^b x(t) d\psi(t)$ exists in the Riemann-Stieltjes sense. Using the Lagrange interpolation polynomial which coincides with $x(t)$ at the points $\{t_{in}\}$ $(i = 1, 2, \cdots, n)$, we construct the so-called mechanical quadratures formula of Gauss' type

$$\int_a^b x(t)\, d\psi(t) = \sum_{i=1}^n H_{in} x(t_{in}) + R_n(x), \tag{15}$$

$$H_{in} = \int_a^b \frac{\Phi_n(t)\, d\psi(t)}{(t - t_{in})\, \Phi_n'(t_{in})}$$

having the property $R_n(t^k) = 0$ $(k = 0, 1, \cdots, 2n-1)$. If (a, b) is finite, it is well known that for $\psi(t) \equiv \int_a^t \frac{dt}{\pi \sqrt{(t-a)(b-t)}}$, which gives rise to the so-called trigonometric polynomials $\cos n \arccos [(a + b - 2t)/(a - b)]$, the Côtes numbers H_{in} are equal for each n: $H_{in} = H_n$ $(i = 1, 2, \cdots, n; n = 1, 2, \cdots)$, and the Gauss formula (15) is therefore, in this special case, at the same time of Tchebichef's type.[24] The question naturally arises whether the two types of formulas coincide in any other case. We propose to show that *the two formulas cannot coincide in any other case.*[25]

Our method is to use the hypothesis of coincidence to derive conditions on the moments $\{\alpha_k\}$ $(k = 0, 1, 2, \cdots)$ of $\psi(t)$, and by this means to show that $\psi(t)$ is uniquely determined in the class of admissible characteristic functions. In

[23] For the basic theory used throughout this section, see J. Shohat, *Théorie générale des polynomes orthogonaux de Tchebichef*, loc. cit.

[24] Mechanical quadratures formulas characterized by the property of possessing equal Côtes numbers were first discussed by P. Tchebichef, Journal de Mathématiques, vol. 19 (1874), pp. 19–34.

[25] The above result was obtained by the author late in 1933 and presented to the American Mathematical Society in November, 1934. I learned recently from Professor Shohat that Professor M. Krawtchouk in June 1934 presented to the All Russian Mathematical Congress in Leningrad a similar theorem, but only for the special case $\psi(x) \equiv \int p(x) dx$.

Cf. M. Krawtchouk, *Sur une question algébrique dans le problème des moments*, Journal de l'Institut Mathématique de l'Académie des Sciences de l'Ukraine, vol. 2 (1934), pp. 87–92; in Ukrainian.

fact, since $[\Phi_n(s) - \Phi_n(t)]/(s - t)$ is a polynomial of degree $n - 1$ in t, if the formula (15) is of the Gauss-Tchebichef type, we must have[26]

$$\int_a^b \frac{\Phi_n(s) - \Phi_n(t)}{s - t}\, d\psi(t) = H_n \sum_{i=1}^{n} \frac{\Phi_n(s) - \Phi_n(t_{in})}{s - t_{in}} \tag{16}$$

$$= H_n \Phi_n(s) \sum_{i=1}^{n} \frac{1}{s - t_{in}} = H_n \Phi_n'(s).$$

Replacing $\Phi_n(s)$, $\Phi_n(t)$, $\Phi_n'(s)$ in (16) by their explicit expressions in terms of the coefficients $\{p_{in}\}$ of $\Phi_n(t)$ and performing the indicated operations, we conclude immediately that for $n = 1, 2, \cdots$,

$$\begin{cases} n\alpha_1 + \alpha_0 p_{1n} = 0, \\ n\alpha_2 + np_{1n}\alpha_1 + 2p_{2n}\alpha_0 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ n\alpha_{n-1} + np_{1n}\alpha_{n-2} + \cdots + (n-1)\, p_{n-1,n}\alpha_0 = 0. \end{cases} \tag{17}$$

To these relations we may add

$$\alpha_n + p_{1n}\alpha_{n-1} + \cdots + p_{nn}\alpha_0 = 0 \qquad (n \geqq 1), \tag{18}$$

$$\alpha_{n+1} + p_{1n}\alpha_n + \cdots + p_{nn}\alpha_1 = 0 \qquad (n \geqq 2), \tag{19}$$

in view of the orthogonality properties of $\Phi_n(t)$. The equations (17), together with (18), give the coefficients $\{p_{kn}\}$ $(k = 1, 2, \cdots, n)$ in terms of $\alpha_0, \alpha_1, \cdots, \alpha_n$, since the determinant of the system is clearly different from zero; therefore, in virtue of (19), α_{n+1} $(n \geqq 2)$ is determined by $\alpha_0, \alpha_1, \cdots, \alpha_n$. That is, the moments $\{\alpha_k\}$ $(k = 0, 1, 2, \cdots)$ are completely determined by $\alpha_0, \alpha_1, \alpha_2$, through a recurrence formula which is independent of (a, b) and $\psi(t)$.

We have only one more step. There is no loss of generality in assuming $\alpha_0 = 1$; by the Schwarz inequality, it follows that

$$\alpha_1^2 \leqq \alpha_2. \tag{20}$$

Moreover, (20) is precisely the necessary and sufficient condition for the existence of real numbers c, d such that $\alpha_1 = \frac{1}{2}(c + d)$, $\alpha_2 = (c - d)^2/8 + (c + d)^2/4$. Inasmuch as all the moments $\{\alpha_k\}$ are determined by α_1 and α_2, it follows that the set $\{\alpha_k\}$ is identical with the set of moments associated with the characteristic function $\bar{\psi}(t) \equiv \int_c^t \frac{dt}{\pi\sqrt{(t-c)(d-t)}}$ in the finite interval (c, d), for the moments of this latter function satisfy the same relations as the set $\{\alpha_k\}$, and

[26] This interesting formula shows that, when coincidence takes place, the numerators $\{\Omega_n(t)\}$ of the successive convergents of the associated continued fraction (Shohat, loc. cit., p. 12) are simply the derivatives of the polynomials themselves. J. Sherman, *Stieltjes continued fractions*, Transactions of the American Mathematical Society, vol. 35 (1933), pp. 64–87, p. 81 has pointed out that this cannot occur with the classical polynomials of Jacobi, Laguerre and Hermite except in the trigonometric case.

$\int_c^d d\bar{\psi}(t) = 1$, $\int_c^d t d\bar{\psi}(t) = \frac{1}{2}(c + d)$, $\int_c^d t^2 d\bar{\psi}(t) = (c - d)^2/8 + (c + d)^2/4$, as can easily be verified. But *the set of moments* $\{\alpha_k\}$ $(k = 0, 1, 2, \cdots)$ *associated with a characteristic function* $\psi(t)$ *in a finite interval* (a, b) *cannot be generated by a different characteristic function in the same or any other interval, finite or infinite.*[27] It follows that coincidence cannot take place in the infinite interval, and that in the finite interval (a, b) the known solution $(c, d) \equiv (a, b)$,

$$\psi(t) \equiv \int_a^t \frac{dt}{\pi \sqrt{(t - a)(b - t)}}$$

is unique.

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[27] As Prof. Shohat pointed out to the author, this fact is an immediate consequence of theorems due to Stieltjes and Carleman on the determinateness of the moment problem. Cf. J. Shohat, loc. cit., p. 7.